Smash-nilpotent cycles on abelian 3-folds

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Abstract. We show that homologically trivial algebraic cycles on a 3-dimensional abelian variety are smash-nilpotent.

Introduction

Let X be a smooth projective variety over a field k. An algebraic cycle Z on X (with rational coefficients) is *smash-nilpotent* if there exists $n > 0$ such that Z^n is rationally equivalent to 0 on X^n. Smash-nilpotent cycles have the following properties:

(1) The sum of two smash-nilpotent cycles is smash-nilpotent.
(2) The subgroup of smash-nilpotent cycles forms an ideal under the intersection product as $(x \cdot y) \times (x \cdot y) \cdots \times (x \cdot y) = (x \times x \times \cdots \times x) \cdot (y \times y \times \cdots \times y)$.
(3) On an abelian variety, the subgroup of smash-nilpotent cycles forms an ideal under the Pontryagin product as $(x * y) \times (x * y) \times \cdots \times (x * y) = (x \times x \times \cdots \times x) * (y \times y \times \cdots \times y)$ where $*$ denotes the Pontryagin product.

Voevodsky [11, Cor. 3.3] and Voisin [12, Lemma 2.3] proved that any cycle algebraically equivalent to 0 is smash-nilpotent. On the other hand, because of cohomology, any smash-nilpotent cycle is numerically equivalent to 0; Voevodsky conjectured that the converse is true [11, Conj. 4.2].

This conjecture is open in general. The main result of this note is:

Theorem 1. *Let A be an abelian variety of dimension ≤ 3. Any homologically trivial cycle on A is smash-nilpotent.*

In characteristic 0 we can improve "homologivally trivial" to "numerically trivial", thanks to Lieberman's theorem [7].

Nori's results in [8] give an example of a group of smash-nilpotent cycles which is not finitely generated modulo algebraic equivalence. The proof of Theorem 1 actually gives the uniform bound 21 for the degree

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of smash-nilpotence on this group, see Remark 2. See Proposition 2 for partial results in dimension 4.

1. Beauville's decomposition, motivically

For any smooth projective variety X and any integer $n \geq 0$, we write as in [1] $CH^n_{\mathbf{Q}}(X) = CH^n(X) \otimes \mathbf{Q}$, where $CH^n(X)$ is the Chow group of cycles of codimension n on X modulo rational equivalence.

Let A be an abelian variety of dimension g. For $m \in \mathbf{Z}$, we denote by $\langle m \rangle$ the endomorphism of multiplication by m on A, viewed as an algebraic correspondence. In [1], Beauville introduces an eigenspace decomposition of the rational Chow groups of A for the actions of the operators $\langle m \rangle$, using the Fourier transform. Here is an equivalent definition: in the category of Chow motives with rational coefficients, the endomorphism $1_A \in \mathrm{End}(h(A)) = CH^g_{\mathbf{Q}}(A \times A)$ is given by the class of the diagonal Δ_A. We have the canonical Chow-Künneth decomposition of Deninger-Murre

$$1_A = \sum_{i=0}^{2g} \pi_i$$

[4, Th. 3.1], where the π_i are orthogonal idempotents and π_i is characterised by $\pi_i \langle m \rangle^* = m^i \pi_i$ for any $m \in \mathbf{Z}$. This yields a canonical Chow-Künneth decomposition of the Chow motive $h(A)$ of A:

$$h(A) = \bigoplus_{i=0}^{2g} h^i(A), \quad h^i(A) = (A, \pi_i)$$

(see [10, Th. 5.2]). Then, under the isomorphism

$$CH^n_{\mathbf{Q}}(A) = \mathrm{Hom}(\mathbb{L}^n, h(A))$$

(where $\mathbb{L}$ is the Lefschetz motive) we have

$$CH^n(A)_{[r]} := \{x \in CH^n_{\mathbf{Q}}(A) \mid \langle m \rangle^* x = m^r x \ \forall m \in \mathbf{Z}\} = \mathrm{Hom}(\mathbb{L}^n, h^r(A)).$$

Remark 1. In Beauville's notation, we have

$$CH^n(A)_{[r]} = CH^n_{2n-r}(A).$$

We shall use his notation in §3.

2. Skew cycles on abelian varieties

Let $\beta \in CH^*_{\mathbf{Q}}(A)$. Assume that $\langle -1 \rangle^* \beta = -\beta$: we say that β is *skew*. This implies that β is homologically equivalent to 0.

For $g \leq 2$, the Griffiths group of A is 0 and there is nothing to prove. For $g = 3$, the Griffiths group of A is a quotient of $CH^2(A)_{[3]}$ [1, Prop. 6]; thus Theorem 1 follows from the more general

Proposition 1. *Any skew cycle on an abelian variety is smash-nilpotent.*

This applies notably to the Ceresa cycle [3], for any genus.

Proof. We may assume β homogeneous, say, $\beta \in CH^n_{\mathbf{Q}}(A)$. View β as a morphism $\mathbb{L}^n \to h(A)$ in the category of Chow motives. Thus, for all i:

$$-\pi_i\beta = \pi_i\langle -1\rangle^*\beta = (-1)^i\pi_i\beta$$

hence $\pi_i\beta = 0$ for i even.

This shows that β factors through a morphism

$$\tilde{\beta} : \mathbb{L}^n \to h^{odd}(A)$$

with $h^{odd}(A) = \bigoplus_{i \text{ odd}} h^i(A)$.

But $\mathbb{L}^n$ is evenly finite-dimensional and $h^{odd}(A)$ is oddly finite-dimensional in the sense of S.-I. Kimura. (Indeed, $S^{2g+1}(h^1(A)) = h^{2g+1}(A) = 0$ by [9, Theorem], and a direct summand of an odd tensor power of an oddly finite-dimensional motive is oddly finite dimensional by [6, Prop. 5.10 p. 186].) Hence the conclusion follows from [6, prop. 6.1 p. 188]. $\square$

Remark 2. Kimura's proposition 6.1 shows in fact that all $z \in \mathrm{Hom}(\mathbb{L}^n, h^{odd}(A))$ verify $z^{\otimes N+1} = 0$ for a fixed N, namely, the sum of the odd Betti numbers of A. If $z \in \mathrm{Hom}(\mathbb{L}^n, h^i(A))$ for some odd i, then we may take for N the i-th Betti number of A. Thus, for $i = 3$ and if A is a 3-fold, we find that all $z \in \mathrm{Hom}(\mathbb{L}, h^3(A))$ verify $z^{\otimes 21} = 0$.

3. The 4-dimensional case

Proposition 2. *If $g = 4$, homologically trivial cycles on A, except perhaps those which occur in parts $CH^2_0(A)$ or $CH^3_2(A)$ of the Beauville decomposition, are smash-nilpotent.*

Proof. Let A be an abelian variety and let $\hat{A}$ denote its dual abelian variety. We know, from [1], the following:

1. $CH^p_s(A) = 0$ for $p \in \{0, 1, g-2, g-1, g\}$ and $s < 0$. [1, Prop. 3a].
2. $CH^p_p(A)$ and $CH^g_s(A)$ consist of cycles algebraically equivalent to 0 for all values of p and all values of $s > 0$. [1, Prop. 4].

For $g = 4$, using these results and Proposition 1 one can conclude smash nilpotence for homologically trivial cycles which are not in $CH^2_0(A)$ or $CH^3_2(A)$. Note that, with the notation of §1,

$$CH^3_2(A) = \mathrm{Hom}(\mathbb{L}^3, h^4(A)), \quad CH^2_0(A) = \mathrm{Hom}(\mathbb{L}^2, h^4(A)).$$

In the case of $CH_0^2(A)$, the problem is whether there are any homologically trivial cycles: in view of the above expression, this is conjecturally not the case, cf. [5, Prop. 5.8]. □

Remark 3. Some of these arguments also follow from a paper of Bloch and Srinivas [2].

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